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LETTER TO THE EDITOR

On the integrability of a new discrete nonlinear Schrödinger equation

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Abstract
We consider the nonlinear Schrödinger equation on the lattice introduced by Leon and Manna two years ago to describe the slowly varying envelope approximation of some nonlinear differential difference equations. We show that this equation does not admit local generalized symmetries of order greater than three. In such a way we prove that the Leon and Manna discrete nonlinear Schrödinger equation does not have the same integrability properties as the Toda lattice equation, from which it has been derived. At the end we provide some reasoning to justify the result obtained.

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1. Introduction

Recently, the reductive perturbation technique [1] has been applied by Leon and Manna [2] to the multiscale analysis of discrete nonlinear evolution equations.

The main results obtained by the reductive perturbation technique in the case of partial differential equations have been the derivation from a given system of simplified equations, while preserving its main properties. The reduced nonlinear equations are fundamentally simpler than the original ones and they generically represent the long-time coherent behaviour of the initial system. Among other results, this procedure has been essential to show the universality character of the nonlinear Schrödinger equation [3].

The situation is quite different in the case of equations on the lattice, when some of variables evolve on a discrete space. The necessity of carrying out the reduction procedure by introducing asymptotic continuous variables, implies that in most of the cases one will get a limiting nonlinear partial differential equation of the kind of the nonlinear Schrödinger equation.

Leon and Manna propose in [2] a set of tools for the multiscale analysis of discrete models which allow one to obtain, under appropriate boundary conditions, a reduced discrete nonlinear equation. These tools rely mainly on the definition of a rescaled large grid and the requirement of a corresponding rescaling of the difference operator. In this way, through the reductive perturbation analysis of the Toda lattice equation

$$\frac{\mathrm{d}^2 x_m}{\mathrm{d}\tau^2} = \mathrm{e}^{x_{m+1}-x_m} - \mathrm{e}^{x_m - x_{m-1}} \tag{1}$$

they obtain the following discrete nonlinear Schrödinger equation:

$$\alpha\psi_{n,tt} = \mathrm{i}\beta(\psi_{n+1} - \psi_{n-1}) + 2|\psi_n|^2\psi_n \tag{2}$$

with α and β nonzero constant real coefficients. The function $\psi_n(t)$ is the lowest-order term in the Fourier expansion of the amplitude of the function $x_m(\tau)$ written in terms of the slowly varying variables t and n.

As $\psi_n(t)$ is a complex function, equation (2) can be written as a system in terms of $u_n = \psi_n$, $v_n = \bar{\psi}_n$:

$$\begin{aligned} u_{n,tt} &= \mathrm{i}\frac{\beta}{\alpha}(u_{n+1} - u_{n-1}) + \frac{2}{\alpha}u_n^2 v_n \\ v_{n,tt} &= -\mathrm{i}\frac{\beta}{\alpha}(v_{n+1} - v_{n-1}) + \frac{2}{\alpha}v_n^2 u_n \end{aligned}$$

where $\bar{\psi}_n$ is the complex conjugate of the function ψ_n.

From the viewpoint of the problem of the existence of generalized symmetries, the time t can be considered to be complex and thus, for any α and β different from zero, rescaling u_n, v_n and t by complex numbers, we can pass to an equivalent and simpler equation:

$$\begin{aligned} u_{n,tt} &= u_{n+1} - u_{n-1} + u_n^2 v_n \\ v_{n,tt} &= -(v_{n+1} - v_{n-1}) + v_n^2 u_n. \end{aligned} \tag{3}$$

This system will be the object of our investigation.

In the following, we are going to show that equation (3) cannot have local generalized symmetries of a high enough order at difference from all well known integrable lattice equations like the Toda, Volterra or discrete nonlinear Schrödinger lattice equations (see e.g. [4] and the review article [5]).

Let us consider the class of equations

$$\begin{aligned} u_{n,tt} &= u_{n+1} + \varphi(u_n, v_n) + \hat{\varphi}(u_{n-1}, v_{n-1}) \\ v_{n,tt} &= -v_{n+1} + \psi(u_n, v_n) + \hat{\psi}(u_{n-1}, v_{n-1}) \end{aligned} \tag{4}$$

depending on four arbitrary functions of two variables which, to contain equation (3), must satisfy the following requirements:

$$\frac{\partial\varphi}{\partial u_n} \neq \text{const} \qquad \frac{\partial\varphi}{\partial v_n} \neq \text{const} \tag{5}$$

$$\frac{\partial\psi}{\partial u_n} \neq \text{const} \qquad \frac{\partial\psi}{\partial v_n} \neq \text{const}. \tag{6}$$

The results which will be presented in section 3 are valid for all equations of the class (4) and, moreover, only require that condition (5) is satisfied.

Section 2 is devoted to the formulation of the problem while in section 3 a theorem showing that equation (2) has no sufficiently high-order symmetries is presented and proven. In section 4, we analyse the procedure introduced in [2] and make some concluding remarks.

2. The generalized symmetry method for equation (4)

Defining

$$U_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix} \tag{7}$$

equation (4) can be expressed as a vector equation of the form

$$U_{n,tt} = \Phi(U_{n+1}, U_n, U_{n-1}) \qquad \Phi = \begin{pmatrix} f \\ g \end{pmatrix} \tag{8}$$

where

$$f = f(u_{n+1}, u_n, v_n, u_{n-1}, v_{n-1}) = u_{n+1} + \varphi(u_n, v_n) + \hat{\varphi}(u_{n-1}, v_{n-1})$$
$$g = g(v_{n+1}, u_n, v_n, u_{n-1}, v_{n-1}) = -v_{n+1} + \psi(u_n, v_n) + \hat{\psi}(u_{n-1}, v_{n-1}).$$

We will look for local generalized symmetries for equation (8) of the form

$$U_{n,\lambda} = \Psi(U_{n+k}, U_{n+k,t}, U_{n+k-1}, U_{n+k-1,t}, \ldots, U_{n+k'}, U_{n+k',t}) \qquad (k > k') \tag{9}$$

as flows commuting with (8) (higher t-derivatives in the rhs of equation (9) are not necessary as they can be expressed via U_{n+i}, $U_{n+i,t}$, using (8)). We assume that Ψ depends essentially on U_{n+k} or $U_{n+k,t}$ and on $U_{n+k'}$ or $U_{n+k',t}$. Differentiating (8) wrt λ, we can write down the compatibility condition: $D_t^2\Psi = D_\lambda\Phi$ (here and in the following D_t and D_λ are the total derivatives wrt t and λ).

To be able to implement the generalized symmetry method (see e.g. [6, 7] and review articles [5, 8–10]), we rewrite (8) in evolutionary form, introducing the new dependent variable V_n: $U_{n,t} = V_n$. Then, defining

$$W_n = \begin{pmatrix} U_n \\ V_n \end{pmatrix} = \begin{pmatrix} u_n \\ v_n \\ \tilde{u}_n \\ \tilde{v}_n \end{pmatrix} \qquad \tilde{u}_n = u_{n,t} \qquad \tilde{v}_n = v_{n,t} \tag{10}$$

equation (8) reads

$$W_{n,t} = F(W_{n+1}, W_n, W_{n-1}) \tag{11}$$

where

$$F = \begin{pmatrix} V_n \\ \Phi \end{pmatrix} = \begin{pmatrix} \tilde{u}_n \\ \tilde{v}_n \\ u_{n+1} + \varphi(u_n, v_n) + \hat{\varphi}(u_{n-1}, v_{n-1}) \\ -v_{n+1} + \psi(u_n, v_n) + \hat{\psi}(u_{n-1}, v_{n-1}) \end{pmatrix} \tag{12}$$

i.e. this is now an evolutionary system of four equations for the four dependent variables u_n, v_n, $\tilde{u}_n$, $\tilde{v}_n$.

The symmetry is rewritten, consequently, as

$$W_{n,\lambda} = G(W_{n+m}, W_{n+m-1}, \ldots, W_{n+m'}) \qquad G = \begin{pmatrix} \Psi \\ \widetilde{\Psi} \end{pmatrix} \tag{13}$$

where $\widetilde{\Psi} = D_t\Psi$. In equation (13), $m = k$ if Ψ does not depend on $U_{n+k,t}$ and $m = k+1$ in the opposite case. Indeed, if Ψ depends on $U_{n+k,t}$, then $\widetilde{\Psi}$ contains $U_{n+k,tt}$ which, due to equation (8), introduces terms containing W_{n+k+1}. For m', there are also two possibilities: in this case we have either $m' = k'$ or $m' = k' - 1$. If we consider the compatibility condition between (11) and (13), i.e. $D_\lambda W_{n,t} - D_t W_{n,\lambda} = D_\lambda F - D_t G = 0$, we obtain $D_\lambda\Phi = D_t^2\Psi$.

Consequently, generalized symmetries for equation (11) and for equation (8) are completely equivalent.

Below we will consider only the situation when a symmetry (9) of order $k \geqslant 4$ exists (the case with right order $k' \leqslant -4$ is completely equivalent; in this case one has only to consider formal series expressed in terms of positive powers of the shift operator T instead of negative ones, cf. (18)). We consider the evolutionary form (11) of the symmetries so as to be able to consider the so-called formal symmetries, i.e. approximate solutions L of the Lax equation

$$L_{,t} = [F_*, L]. \tag{14}$$

Here F_* is the Frechet derivative of F:

$$F_* = F_n^{(1)} T + F_n^{(0)} + F_n^{(-1)} T^{-1}, \tag{15}$$

where $F_n^{(i)}$ are rank-4 matrices obtained as partial derivatives of F, given by $F_n^{(i)} = \partial F / \partial W_{n+i}$. For convenience, it is simpler to write them as 2×2 blocks,

$$F_n^{(\pm 1)} = \begin{pmatrix} 0 & 0 \\ \Phi_n^{(\pm 1)} & 0 \end{pmatrix} \qquad F_n^{(0)} = \begin{pmatrix} 0 & E \\ \Phi_n^{(0)} & 0 \end{pmatrix} \tag{16}$$

where E is the unit matrix, and the matrices $\Phi_n^{(i)}$ $(i = -1, 0, 1)$ are defined as the partial derivatives of Φ:

$$\Phi_n^{(i)} = \frac{\partial \Phi}{\partial U_{n+i}} = \begin{pmatrix} f_{,u_{n+i}} & f_{,v_{n+i}} \\ g_{,u_{n+i}} & g_{,v_{n+i}} \end{pmatrix}$$

i.e.

$$\begin{gathered} \Phi_n^{(-1)} = \begin{pmatrix} \hat{\varphi}_{,u_{n-1}} & \hat{\varphi}_{,v_{n-1}} \\ \hat{\psi}_{,u_{n-1}} & \hat{\psi}_{,v_{n-1}} \end{pmatrix} \qquad \Phi_n^{(0)} = \begin{pmatrix} \varphi_{,u_n} & \varphi_{,v_n} \\ \psi_{,u_n} & \psi_{,v_n} \end{pmatrix} \\ \Phi_n^{(1)} = \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{gathered} \tag{17}$$

The solution L of (14) can be written as a formal series in decreasing powers of T whose coefficients are matrices of rank 4:

$$L = L_n^{(m)} T^m + L_n^{(m-1)} T^{m-1} + \dots \qquad L_n^{(m)} \neq 0 \tag{18}$$

where m defines the order of the solution.

If we introduce

$$\Omega(L) = L_{,t} - [F_*, L] \tag{19}$$

we can see that, for L of order m, the series $\Omega(L)$ is in general of order $m+1$:

$$\Omega(L) = \Omega_n^{(m+1)} T^{m+1} + \Omega_n^{(m)} T^m + \dots.$$

The series representation (18) of L is a formal symmetry of order m and length l if the l highest coefficients $\Omega_n^{(i)}$ $(i = m+1, m, \dots, m-l+2)$ of $\Omega(L)$ vanish, i.e. the series representation (18) is an approximate solution of (14) with l correct coefficients. One can see immediately, using equation (19), that formal symmetries (unlike generalized symmetries) can be multiplied by each other

$$\Omega(L\hat{L}) = \Omega(L)\hat{L} + L\Omega(\hat{L}).$$

Using this result, the length of the product can be easily found. For example, if $\det L_n^{(m)} \neq 0$, then the formal symmetry $(L)^2$ has order $2m$ and length l (the same as L).

One can also easily show that the existence of a generalized symmetry (13) implies the existence of a formal symmetry. Indeed, considering the Frechet derivatives of G, given by

$$G_* = \sum_{i=m'}^{m} \frac{\partial G}{\partial W_{n+i}} T^i$$

and using the relations

$$(G_{,t})_* = (G_*)_{,t} + G_*F_* \qquad (F_{,\lambda})_* = (F_*)_{,\lambda} + F_*G_*$$

one can calculate the Frechet derivative of the compatibility condition $G_{,t} = F_{,\lambda}$. We have the following result:

$$\Omega(G_*) = (G_*)_{,t} - [F_*, G_*] = (F_*)_{,\lambda}.$$

This shows that G_* is a formal symmetry of order m and length m (i.e. in this case $l = m$).

Thus, starting from a symmetry of equation (8) of order $k \geqslant 4$, we can pass to a symmetry of equation (11) of order $m \geqslant 4$, and are led to a formal symmetry L of order $m \geqslant 4$ and length $l \geqslant 4$. In section 3 we will see that we are not able to calculate the first four coefficients of L for systems of the form (4), (5). This implies that those systems cannot have local generalized symmetries of order greater or equal to 4.

We restrict ourselves to symmetries (9), (13) with no explicit dependence on the discrete variable n and on the continuous time t. Under such conditions, the rhs of (13) only depends on functions W_{n+i} $(i = m, m-1, \ldots, m')$. This is a very strong assumption, true for most local generalized symmetries of integrable equations on the lattice. As no explicit dependence on t and n is allowed, first-order linear difference equations have only constant solutions. Moreover, we will assume, as is usually done in the generalized symmetry method, that the functions W_{n+i}, as well as their components u_{n+i}, v_{n+i}, $\tilde{u}_{n+i}$, $\tilde{v}_{n+i}$ (see (10)), for any i are independent variables.

Let us end this section by enumerating some important properties of solutions of first order scalar difference equations for functions h, depending just on W_{n+i}. There are only two possibilities for h: either $h = 0$ or $h = \text{const}$ identically for any n (this would not be true in the case of n-dependent functions, see [7]). The following two statements can be easily proved, using the dependence of h on the independent variables W_{n+i}:

$$\text{if} \quad h \in \operatorname{Ker}(T-1) \quad \Rightarrow \quad h = \text{const}(a) \tag{A}$$

$$\text{if} \quad h \in \operatorname{Ker}(T+1) \quad \Rightarrow \quad h = 0. \tag{B}$$

If the function h depends only on W_n, then

$$\text{if} \quad h(W_n) \in \operatorname{Im}(T-1) \quad \Rightarrow \quad h = 0 \tag{C}$$

$$\text{if} \quad h(W_n) \in \operatorname{Im}(T+1) \quad \Rightarrow \quad h = \text{const}. \tag{D}$$

Moreover one can prove that if $D_t(h) = 0$, then $h = \text{const}$.

3. Main theorem

It is easier to calculate the matrix coefficients of the formal symmetry L (18), considering them as 2×2 matrix blocks which will be denoted as

$$L_n^{(i)} = \begin{pmatrix} A_n^{(i)} & B_n^{(i)} \\ C_n^{(i)} & D_n^{(i)} \end{pmatrix}.$$

Collecting in equation (14) the coefficients of T^{m+1} (i.e. requiring that $\Omega_n^{(m+1)} = 0$), one obtains

$$F_n^{(1)} L_{n+1}^{(m)} = L_n^{(m)} F_{n+m}^{(1)}.$$

As $\det \Phi_n^{(1)} \neq 0$, the equation for $L_n^{(m)}$ is equivalent to two 2×2 matrix equations:

$$B_n^{(m)} = 0 \tag{20}$$

$$\Phi_n^{(1)} A_{n+1}^{(m)} = D_n^{(m)} \Phi_{n+m}^{(1)}. \tag{21}$$

Now, taking into account equation (20), let us write down the equations obtained by requiring that length l be $l \geqslant 3$, i.e. by requiring that the coefficients of T^m, T^{m-1} in $\Omega(L)$ be also zero. The coefficients of the power T^m, written in block form, satisfy the following equations:

$$D_n^{(m)} = A_n^{(m)} \tag{22}$$

$$A_{n,t}^{(m)} = C_n^{(m)} - B_n^{(m-1)}\Phi_{n+m-1}^{(1)} \tag{23}$$

$$D_{n,t}^{(m)} = -C_n^{(m)} + \Phi_n^{(1)} B_{n+1}^{(m-1)} \tag{24}$$

$$C_{n,t}^{(m)} = \Phi_n^{(0)} A_n^{(m)} - D_n^{(m)}\Phi_{n+m}^{(0)} + \Phi_n^{(1)} A_{n+1}^{(m-1)} - D_n^{(m-1)}\Phi_{n+m-1}^{(1)}. \tag{25}$$

Considering the coefficients of the power T^{m-1}, one obtains

$$B_{n,t}^{(m-1)} = D_n^{(m-1)} - A_n^{(m-1)} \tag{26}$$

$$A_{n,t}^{(m-1)} = C_n^{(m-1)} - B_n^{(m-2)}\Phi_{n+m-2}^{(1)} - B_n^{(m-1)}\Phi_{n+m-1}^{(0)} \tag{27}$$

$$D_{n,t}^{(m-1)} = -C_n^{(m-1)} + \Phi_n^{(1)} B_{n+1}^{(m-2)} + \Phi_n^{(0)} B_n^{(m-1)}. \tag{28}$$

Using equations (20)–(28), we can now prove the following theorem:

Theorem. Systems of the form (4), (5) cannot have local generalized symmetries (9) with $k \geqslant 4$.

Proof. If such a symmetry exists, then there should also exist a formal symmetry (18) of the system (11) of order $m \geqslant 4$ and length $l \geqslant 4$. Defining

$$A_n^{(i)} = \begin{pmatrix} a_n^{(i)} & b_n^{(i)} \\ c_n^{(i)} & d_n^{(i)} \end{pmatrix} \qquad B_n^{(i)} = \begin{pmatrix} \alpha_n^{(i)} & \beta_n^{(i)} \\ \gamma_n^{(i)} & \delta_n^{(i)} \end{pmatrix} \tag{29}$$

where $a_n^{(i)}, b_n^{(i)}, c_n^{(i)}, d_n^{(i)}, \alpha_n^{(i)}, \beta_n^{(i)}, \gamma_n^{(i)}$ and $\delta_n^{(i)}$ are scalar quantities, equations (21), (22) imply $\Lambda A_{n+1}^{(m)} = A_n^{(m)}\Lambda$ (see (17)), and give that $a_n^{(m)}, d_n^{(m)} \in \mathrm{Ker}\,(T-1)$ and $b_n^{(m)}, c_n^{(m)} \in \mathrm{Ker}\,(T+1)$.

Taking into account the results (A) and (B), we get that the matrix $A_n^{(m)}$ is constant and diagonal:

$$A_n^{(m)} = A = \mathrm{diag}(a, d). \tag{30}$$

Equations (23), (24) and (30) give

$$C_n^{(m)} = \Lambda B_{n+1}^{(m-1)} = B_n^{(m-1)}\Lambda. \tag{31}$$

Hence matrices $B_n^{(m-1)}$, $C_n^{(m)}$ are also constant and diagonal:

$$B_n^{(m-1)} = B = \mathrm{diag}(\alpha, \delta) \qquad C_n^{(m)} = C = \mathrm{diag}(\alpha, -\delta). \tag{32}$$

Let us now prove that

$$d = (-1)^m a. \tag{33}$$

From, equation (26) we get $D_n^{(m-1)} = A_n^{(m-1)}$, and consequently equation (25) takes the form

$$\Phi_n^{(0)} A - A\Phi_{n+m}^{(0)} + \Lambda A_{n+1}^{(m-1)} - A_n^{(m-1)}\Lambda = 0. \tag{34}$$

The elements of the right upper corner of equation (34) provide us with the equation

$$d\varphi_{,v_n} - aT^m(\varphi_{,v_n}) + b_{n+1}^{(m-1)} + b_n^{(m-1)} = 0$$

i.e.

$$(d - aT^m)(\varphi_{,v_n}) \in \mathrm{Im}\,(T+1). \tag{35}$$

As

$$(T+1)\sum_{i=0}^{m-1}(-1)^i T^i = 1 - (-1)^m T^m$$

we can write

$$T^m = (-1)^m - (T+1)\sum_{i=0}^{m-1}(-1)^{i+m}T^i$$

and consequently condition (35) can be rewritten as

$$(d - (-1)^m a)\varphi_{,v_n} \in \mathrm{Im}\,(T+1).$$

The function in the lhs of this condition depends only on W_n, and therefore, due to condition (D), must be a constant. But $\varphi_{,v_n} \neq \text{const}$ and consequently equation (33) must be true.

To get further constraints let us take into account that if a formal symmetry exists, then its powers give new formal symmetries. Let us consider

$$(L)^2 = \tilde{L}_n^{(2m)}T^{2m} + \tilde{L}_n^{(2m-1)}T^{2m-1} + \ldots.$$

As we have proven before,

$$L_n^{(m)} = \begin{pmatrix} A & 0 \\ C & A \end{pmatrix} \tag{36}$$

and consequently

$$\tilde{L}_n^{(2m)} = L_n^{(m)}L_{n+m}^{(m)} = \begin{pmatrix} A^2 & 0 \\ 2AC & A^2 \end{pmatrix}. \tag{37}$$

If $A = 0$, then:

$$\begin{aligned} L_n^{(m-1)} &= \begin{pmatrix} A_n^{(m-1)} & B \\ C_n^{(m-1)} & A_n^{(m-1)} \end{pmatrix} \\ \tilde{L}_n^{(2m-1)} &= L_n^{(m)}L_{n+m}^{(m-1)} + L_n^{(m-1)}L_{n+m-1}^{(m)} = \begin{pmatrix} BC & 0 \\ H_n & BC \end{pmatrix} \end{aligned} \tag{38}$$

where H_n is a well defined matrix of rank 2. $L_n^{(m)} \neq 0$ and consequently, as $A = 0$, $C \neq 0$. In this case $\tilde{L}_n^{(2m)} = 0$, but $\tilde{L}_n^{(2m-1)} \neq 0$ because $BC = \mathrm{diag}(\alpha^2, -\delta^2) \neq 0$ (see equation (32)). The order of the formal symmetry $(L)^2$ in this case is $2m-1$, and the length is $l-1$. So if $A = 0$, there must exist a formal symmetry $\tilde{L}$ such that $\tilde{A} \neq 0$, so that the formal symmetry has order $\widetilde{m} \geqslant 4$ and length $\tilde{l} \geqslant 3$.

Consider a formal symmetry L with $A \neq 0$ of order $m \geqslant 4$ and length $l \geqslant 3$. It follows from equations (30), (33) that if $A \neq 0$, then $A = a\,\mathrm{diag}(1, (-1)^m)$ with $a \neq 0$. Passing to $(L)^2$, we see from equation (37) that the diagonal blocks have the form $A^2 = a^2E$. The order of such formal symmetry $(L)^2$ becomes $2m$ and the length remains the same. Formal symmetries may be multiplied by nonzero constants with no change of order and length. So we can assert that, if a formal symmetry with $k \geqslant 4$ exists, there must exist a formal symmetry L such that $A = E$, $m \geqslant 4$, $l \geqslant 3$. As $l \geqslant 3$, all conditions (20)–(28) must be satisfied. Then, if we take the left upper corner elements in equation (34) with $A = E$, we obtain

$$(1 - T^m)(\varphi_{,u_n}) + (T-1)(a_n^{(m-1)}) = 0.$$

For any $m \geqslant 1$,

$$1 - T^m = -(T-1)\sum_{i=0}^{m-1}T^i. \tag{39}$$

Hence $(T-1)(a_n^{(m-1)} - \sum_{i=0}^{m-1}T^i(\varphi_{,u_n})) = 0$ and, due to the property (A), we get

$$a_n^{(m-1)} = \text{const} + \sum_{i=0}^{m-1}T^i(\varphi_{,u_n}). \tag{40}$$

As $D_n^{(m-1)} = A_n^{(m-1)}$ and $B_n^{(m-1)} = B$, the sum of equations (27) and (28) gives

$$2A_{n,t}^{(m-1)} = \Lambda B_{n+1}^{(m-2)} - B_n^{(m-2)}\Lambda + \Phi_n^{(0)} B - B\Phi_{n+m-1}^{(0)}. \tag{41}$$

Considering the left upper corner elements of equation (41), we obtain the condition $2a_{n,t}^{(m-1)} = (T-1)(\alpha_n^{(m-2)}) + (1 - T^{m-1})(\alpha\varphi_{,u_n})$. This condition can be written, taking into account equations (39), (40), in the following way:

$$a_{n,t}^{(m-1)} = \sum_{i=0}^{m-1} T^i(\varphi_{,u_n})_{,t} \in \mathrm{Im}\,(T-1). \tag{42}$$

For any $m \geqslant 1$, we have (see (39))

$$T^m = 1 + (T-1)\sum_{i=0}^{m-1} T^i.$$

Hence $a_{n,t}^{(m-1)}$ can be expressed in the form $a_{n,t}^{(m-1)} = m(\varphi_{,u_n})_{,t} + (T-1)(h)$. This allows us to rewrite equation (42) as the following condition:

$$(\varphi_{,u_n})_{,t} = \varphi_{,u_n u_n} u_{n,t} + \varphi_{,u_n v_n} v_{n,t} \in \mathrm{Im}\,(T-1). \tag{43}$$

Now the function in the lhs of equation (43) depends only on the variable W_n; and therefore, due to condition (C), is equal to zero. The components of W_n are independent, and consequently $\varphi_{,u_n u_n} = \varphi_{,u_n v_n} = 0$, i.e. $\varphi_{,u_n} = \text{const}$. This is in contradiction with the assumption (5), and thus this implies that no generalized symmetry of the form (13) with $m \geqslant 4$ or (9) with $k \geqslant 4$ can exist. □

4. Conclusions

We have proven in section 3 that equation (2) has no local generalized symmetry of a high enough order.

The result obtained was in a certain way to be expected. Up to now no integrable equation of the form of equation (2) has been found even if equation (2) has been obtained by reducing the Toda lattice (1). In principle the reduction technique should preserve the integrability property; at least, that is what happens in the continuous case [3]. The fact that this does not happen in the discrete case seems to imply that some of the hypotheses considered by Leon and Manna [2] are not correct. Apart from a few summation identities, the main ansatz considered in [2] rely on the definition of a rescaled large grid N and the requirement of a rescaling of the difference operator by $1/N$.

The introduction of a large rescaled lattice characterized by a lattice spacing N times the original one, where N is a very large number, is a natural renormalization procedure. The form of rescaling of the difference operator considered by [2] is not at all obvious. In fact, it seems to us that it contains some hypothesis which can provide completely unreliable consequences. Let us fix, following Leon and Manna [2], a small parameter ϵ by

$$\epsilon^2 = 1/N. \tag{44}$$

Then for any given m, we can consider the set of points $\{\cdots, m-N, m, m+N, \cdots\}$ of a *large grid* indexed by a *slow variable* n, such that

$$\cdots, (m-N) \to (n-1), m \to n, (m+N) \to (n+1), \cdots. \tag{45}$$

We have to relate the discrete derivatives in the two different grids described by the two different indexes n and m. To that end we define the derivatives in the original variable m as

$$\begin{aligned} \nabla\phi_m &= \phi_{m+1} - \phi_{m-1} \\ \nabla^k\phi_m &= \sum_{\ell=0}^{k}(-1)^\ell \frac{k!}{\ell!(k-\ell)!}\phi_{m+k-2\ell} \end{aligned} \tag{46}$$

and the derivatives in the new variable n defined in (45) as

$$\Delta_N\phi_m = \phi_{m+N} - \phi_{m-N} = \phi_{n+1} - \phi_{n-1}. \tag{47}$$

Then we have the following identity:

$$\Delta_N\phi_m = \sum_{\ell=0}^{q}\frac{(2q+1)(q+\ell)!}{(q-\ell)!(2\ell+1)!}\nabla^{2\ell+1}\phi_m \qquad N = 2q+1. \tag{48}$$

Leon and Manna introduce in [2] the hypothesis of *slow variation* of a generic function ϕ_n through the condition

$$|\nabla^{k+1}\phi_m| = \epsilon^2|\nabla^k\phi_m| + \mathcal{O}(\epsilon^4). \tag{49}$$

Equation(49) is a necessary condition to rewrite $\nabla\phi_m$ in terms of $\Delta_N\phi_m$ via equation (48). One can show very easily that equation (49) gives undetermined results as soon as $N > 3$. In fact, from (49) one has

$$|\nabla^2\phi_m| = \epsilon^2|\nabla\phi_m| + \mathcal{O}(\epsilon^4) \tag{50}$$

$$|\nabla^3\phi_m| = \epsilon^2|\nabla^2\phi_m| + \mathcal{O}(\epsilon^4). \tag{51}$$

Substituting (50) into (51), we get

$$|\nabla^3\phi_m| = \epsilon^2(\epsilon^2|\nabla\phi_m| + \mathcal{O}(\epsilon^4)) + \mathcal{O}(\epsilon^4) = \mathcal{O}(\epsilon^4) \tag{52}$$

which is an undetermined equation.

Thus a new *slow variation* hypothesis is necessary to provide reasonable results. Work on this is in progress.

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